ASTR450 Homework # 7 – The 3-Body Problem
Due Tuesday, April 8

1. A problem similar to Problem 3 from the midterm. a) Orbital Elements. Find the six orbital elements $(a, e, i, \Omega, \omega, \nu)$ for an asteroid which, at time t=0, has $(X, Y, Z) = (2.5AU, 0, 0)$ and $(V_x, V_y, V_z) = (V_E/\sqrt{10}, -V_E/\sqrt{5}, 0)$. Here V_E is the speed of the Earth in its orbit, and an AU is the astronomical unit. Take the reference plane to be the XY plane and the reference direction to be $\hat{X}$. It is easiest to use dimensionless units; take $V_E = 30$km/s to be the unit of velocity, and the AU to be the unit of distance, and $GM = 1$ to define the unit of mass (so in dimensionless units, $V_E = 1, GM = 1$, and the Earth-Sun distance is one).
b) Find the pericenter distance, the apocenter distance (if it exists), and the semilatus rectum of the orbit, and use these quantities, and your orbital elements, to draw a reasonably accurate sketch of the orbit relative to the X and Y axes.

2. An advanced version of Problem 4 from the Midterm. Consider a special case of the circular restricted three body problem where $M_1 >> M_2 >> m$ (mass hierarchy).
a) Find the distances to each of the Lagrange points from the center of mass treating m as a test particle and assuming $R >> x >> \mu R$, where R is the distance from M_1 to M_2, and $\mu = M_2/(M_1+M_2)$. Find x, the radius of the Hill Sphere, first. Your answers should be accurate to order x/R, but you may drop terms of order $(x/R)^2$ and μ^1.
b) Argue that a small displacement from each of the Lagrange Points in the $\hat{z}$ direction will result in small vertical oscillations. Although L_1, L_2, and L_3 are unstable to motion in the XY plane, oscillations are stable in the $\hat{z}$ direction.
c) To the same level of approximation as part a), find the oscillation frequency at each of the five Lagrange points in terms of the mean motion $\Omega = \sqrt{G(M_1 + M_2)/R^3}$ of M_2 about M_1. Interpret your answers physically. Calculate x for the Earth, and integrate some orbits starting near L_1 and L_2 using the Planetary Satellite Integrator. Can you get some orbits to remain near the Lagrange Points and verify your calculated oscillation frequencies? (It's hard - you'll have to calculate the velocity in the inertial frame that makes the particle have no velocity in the rotating frame. Even then, you'll probably only see an oscillation or two.) See the next problem for some hints on using the Planetary Satellite Integrator.

3. Class Research Project. What if the Moon's orbit were different? Use the Planetary Satellite Integrator to investigate distant orbits around the Earth. Use the default settings of the form, but change the satellite's initial conditions, the integration time, and the accuracy parameter as needed. In each case, start the orbit above the positive X-axis (the reference direction) at its maximum height above the XY plane (the reference plane). See the Help File for comments on the coordinate system, and check to see that you've got the initial conditions right by viewing the Orbital Elements vs. Time plot or the Cartesian Coordinates vs. Time Plot.

Chris Allen: Initially Circular Orbits with $i = 0°$
Mike Asbury: Initially Circular Orbits with $i = 30°$
Ray Brown: Initially Circular Orbits with $i = 50°$
Jason Budinoff: Initially Circular Orbits with $i = 70°$
Heather Cohen: Initially Circular Orbits with $i = 90°$
Elbert Macau: Initially Circular Orbits with $i = 110°$
Linda Harden: Initially Circular Orbits with $i = 130°$
Alaa Ibrahim: Initially Circular Orbits with $i = 150°$
Lori Lanier: Initially Circular Orbits with $i = 180°$

Run simulations for different initial distances larger than the Moon's current distance (60 Earth Radii), and determine the fate of orbits: do they remain bound, escape the Earth, or crash into our planet? Where do transitions between the various regimes take place? Note: to determine if an orbit crashes into Earth or not, check to see if i) the integration ends early and ii) the final distance of the satellite is near 1 Earth radius. What is the escape signature in orbital elements? Look at orbits in inertial X,Y,Z space, in coordinates that rotate with the Earth's mean motion (Xsun, Ysun, Z), and in orbital elements to get an understanding of what is happening to each orbit. Do your orbits ever cross the Zero Velocity Curves (ZVCs)? Are the ZVCs a good indicator of the bound-escape transition? Print out examples of interesting orbits to discuss in class on Tuesday. Those of you doing the homeworks regularly, please write up the results of your investigations and attach plots of interesting orbits (with the initial conditions labeled!). If you'd like to explore further, try some eccentric orbits (be sure to note all of the initial orbital elements)!