Building Physical Intuition in Mechanics

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1. INTRODUCTION

When solving physics problems it's easy to get overwhelmed by the complexity of some of the concepts and equations. It's important to have ways to navigate through these complexities and reduce errors. One of the best navigation tools is a sense of what the answer should look like. What units should it have? How should it behave in easily-understood limits? What are the symmetries of the problem? What should the answer depend on? You should check every answer you get against these common-sense guides. This will cut down dramatically on errors in derivation. Even more importantly, it will help build up your intuition about physics, because you will be able to approach problems by constraining the answer first. This is especially good practice for students, but also is helpful in research: both the authors have seen several examples in textbooks and scientific publications that can be shown to be incorrect by simple checks such as these.

To help build intuition and good habits when doing derivations, we have produced a list of problems. Unlike standard problems, in which the idea is to get the correct answer, here we emphasize ruling out wrong answers. For each problem, consider each of the possible answers and decide whether or not the answer can be ruled out by some physical argument. A complete solution to one of these problems, therefore, consists of a discussion of each of the possible answers.

In this document, we have focused on problems involving classical mechanics because the problems tend to be clean and physical insight is often close to everyday experience. The ideas and concepts presented here, however, can be applied to problems from all subfields of physics. Symbols are defined in each problem, and in addition we provide a symbols list at the end of the document.

2. CHECKING UNITS

Units are the first thing to check when considering possible answers to a problem. Any equation that you write must be dimensionally correct. Check your equations occasionally as you go through a derivation. It takes just a second to do so, and you can quickly catch many common errors. Remember this general rule: in all physically valid solutions, the argument of trigonometric functions, exponentials, logs, hyperbolic functions, etc. must be dimensionless. Taking the cosine of something with units of mass or length makes no physical sense. In a similar way, if you are solving for the energy of an oscillation and come up with an answer that has units of momentum, that answer must be wrong.

For each of the problems below, imagine that you and several friends have just gone through lengthy derivations and have come up with a number of different answers. Only one can be right, but how do you know which answers (and derivations) are suspect? Testing the units of the final answer can provide clues - if the dimensions of an answer do not match those of the quantity that you are looking for, then that answer (and the accompanying derivation) are wrong. Checking units will never tell you that an answer must be right, but it can tell you that an answer must be wrong. If a final answer is dimensionally incorrect, intermediate results in a derivation can also be tested; this can often help to rapidly pinpoint the source of the error. Each of the problems in this section has several incorrect answers and only one that might be correct.

Here's an example:

1. A daredevil is shot out of a cannon at speed v and angle θ from horizontal. Earth's gravitational acceleration, g, is assumed constant, and air resistance is neglected. How far downrange, D, does the daredevil fly before hitting the ground?

A) $D = 2v^2 \cos\theta$
B) $D = (2v^2/g)\sin\theta\cos\theta$
C) $D = 2g\sin\theta\cos\theta$
D) $D = 2vg(\cos\theta - \sin\theta)$
E) $D = (2v^2/g)\sin g$

Answer: Distance is measured in meters, velocities in m/s, and acceleration in m/s^2. All of the above answers have left-hand sides which are distances in units of meters, so the correct answer must have units of meters on the right as well.
A) has units of velocity squared (WRONG)
B) has units of meters (COULD BE OK)
C) has units of acceleration (WRONG)
D) has units of meter squared per second squared (WRONG)
E) the argument of the sine has units (WRONG)
Note that units checking like this is important but does have limitations. For example, any equation that is dimensionally correct is also dimensionally correct if either side is multiplied by an arbitrary dimensionless factor.

Dimension checking can be used to get the units of physical constants as in the following example:

2. What are the units of G, the gravitational constant?

A) $\mathrm{m^2 kg^{-1} s^{-3}}$
B) $\mathrm{m^3 kg^{-1} s^{-2}}$

C) $\mathrm{m}^2\mathrm{kg}^{-2}\mathrm{s}^{-1}$
D) $\mathrm{m}^1\mathrm{kg}^2\mathrm{s}^2$
E) $\mathrm{m}^3\mathrm{kg}^{-3}\mathrm{s}^{-1}$

Answer: The easiest way to get these units quickly is to remember any simple correct formula which uses G. For instance, the gravitational acceleration at the Earth's surface is $g = GM/R^2$, where M and R are the Earth's mass and radius. Solve for G: $G = R^2 g/M$. The units of G, therefore, are the same as the units of $R^2 g/M$: $(\mathrm{m}^2)(\mathrm{m/s}^2)/(\mathrm{kg})$ $= \mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$. The correct answer is B). Other formulae involving G would work equally well. How else can you get the units of G quickly on a test?

3. What is the maximum height attained by the daredevil in problem 1?

A) $H = 2v^3 \sin\theta$
B) $H = (2g/v^2)\sin\theta$
C) $H = 2v\sin\theta\cos\theta$
D) $H = 2vg^2(\cos\theta - \sin\theta)$
E) $H = (v^2/g)\sin\theta$
F) $H = (2v^2/g^2)\sin g$

4. The energy flux (energy per unit time) from an area A of the Sun's emitting surface is given by $F = \sigma A T^4$ where T is the temperature. What are the units of the Stefan-Boltzmann constant σ?

A) $\mathrm{W/m^2K^4}$
B) $\mathrm{W/m^3K^4}$
C) $\mathrm{W/m^2K^2}$
D) $\mathrm{W/m^4K^2}$

5. A mass m falls from a height h in a constant gravitational field with acceleration g. The speed of the mass when it hits the ground could be:

A) $v = 2m$
B) $v = 2h$
C) $v = 2g$
D) $v = \sqrt{2gh}$
E) $v = \sqrt{2mg}$
F) $v = \sqrt{2mh}$

6. A particle undergoes a constant acceleration a from rest. What is the distance it travels after time t?

A) $d = \frac{1}{2}a$
B) $d = \frac{1}{2}at$
C) $d = \frac{1}{2}at^2$
D) $d = \frac{1}{2}at^3$

7. Newton's second law $F = ma$ gives the acceleration a experienced by a body of mass m subject to a force F. If r is a distance, and ω is an angular frequency (with units of s^{-1}), which of the following might be a valid force, where v is a speed?

A) $F = mv$
B) $F = mv^2$
C) $F = m\omega^3 r^2/v$
D) $F = m\omega^2$
E) $F = m\omega^2 r^2/v$

8. Let a mass m be in a circular orbit of radius r and angular frequency ω radians per second. What is the centripetal force needed to keep the mass in that orbit?

A) $F = m$
B) $F = r\omega$
C) $F = mr\omega$
D) $F = mr\omega^2$
E) $F = mr^2\omega^2$

9. Pressure is force divided by area. The pressure at the center of Jupiter (a planet with mass M and radius R) is proportional to:

A) GM^2/R^2
B) GM^2/R^3
C) GM^2/R^4
D) GM^2/R^5

10. The energy released during the accretion of Jupiter is proportional to:

A) GM^2/R
B) GM^2/R^2
C) GM/R
D) GM/R^2

11. Consider a spring with a force constant k, so that if the spring is displaced by a distance x from its equilibrium position then the restoring force is $F = kx$. What could be the oscillation frequency of a mass m at the end of the spring?

A) $\omega = k$
B) $\omega = m$
C) $\omega = km$
D) $\omega = k/m$
E) $\omega = \sqrt{km}$
F) $\omega = \sqrt{k/m}$

12. The sound speed c_s in an ideal gas depends only on the pressure P and the density ρ of the gas. The correct expression is:

A) $c_s = \sqrt{\rho/P}$
B) $c_s = \sqrt{P/\rho}$
C) $c_s = P^2/\rho$
D) $c_s = \rho^2/P$
E) $c_s = \sqrt{P\rho}$

13. A wire has a tension S on it; S is measured in units of force. If the wire has a mass per length μ, what is the speed of a transverse wave in the wire?

A) $v = \sqrt{S\mu}$
B) $v = \mu/S$
C) $v = \mu S$
D) $v = \sqrt{S/\mu}$

14. The bulk modulus B of a material is defined as the ratio of change of pressure to fractional change in volume: $B = \Delta P/(\Delta V/V)$. If a material with density ρ has a bulk modulus B, what is the speed of a longitudinal wave in the material?

A) $v = \sqrt{B\rho}$
B) $v = \rho/B$
C) $v = \rho B$
D) $v = \sqrt{B/\rho}$

15. A sound wave with speed c_s is propagated through a medium of density ρ. An instrument measures that the pressure change due to the sound wave is ΔP. What is the intensity measured by the instrument, where intensity is the energy per time per area, $I = dE/dtdA$?

A) $I = \frac{1}{2}\Delta P \rho c_s$
B) $I = \frac{1}{2}\Delta P^2/(\rho c_s)$
C) $I = \frac{1}{2}\rho c_s/\Delta P$
D) $I = \frac{1}{2}\Delta P c_s/\rho$

16. You decide to exercise your vocal talents in the shower. If the shower has length and width l and the speed of sound is c_s, what is the frequency of the note you should hit to get the maximum resonant effect?

A) $\nu = lc_s$
B) $\nu = 1/(lc_s)$
C) $\nu = c_s/l$
D) $\nu = l/c_s$

17. The gravitational freefall time t for a uniform spherical cloud of interstellar gas with radius R and density ρ is:

A) $t = \sqrt{3\pi/(32G\rho^2)}$
B) $t = \sqrt{3R\pi/(32G\rho)}$
C) $t = R\sqrt{3\pi/(32G\rho)}$
D) $t = \sqrt{3\pi/(32G\rho)}$
E) $t = \sqrt{3\pi/(32RG\rho)}$

18. A spring of spring constant k supports a mass m in a gravitational field with constant acceleration g. The spring is initially unstretched. When the mass is released, how far does it fall before it bounces back up?

A) $h = 2mgk$
B) $h = 2mg/k$
C) $h = 2mk/g$
D) $h = 2g/(mk)$

19. The first law of thermodynamics relates the change in heat energy, dQ, to the change in entropy dS (with units of Boltzmann's constant k) and volume dV at a given temperature T and pressure P. Which of the following could be the first law of thermodynamics?

A) $PdQ = TdS - dV$
B) $dQ = PdS + TdV$
C) $TdQ = dS + dV$
D) $dQ = TdS - PdV$

20. A particle moves through a field of spheres of cross section σ and number density n with velocity v. What is its frequency ν of collision with the spheres?

A) $\nu = n\sigma + v$
B) $\nu = n/(\sigma v)$
C) $\nu = n\sigma v$
D) $\nu = 1/(n\sigma v)$

3. CHECKING LIMITS

Dimensional analysis of an expression is a skill that, once mastered, should always be used. It does have serious limitations, however, and cannot, for instance, tell you when you have dropped a factor of 2 or $\cos\theta$. More sophisticated tests are needed in these cases to catch additional errors and to bolster your confidence in answers that passes all tests. One of the most powerful ways to further test your answer is to consider key limiting cases. Always check your final answers and important intermediate results to see if they behave correctly in as many different limiting situations as you can imagine. Sometimes you will know how a general expression should behave if a particular variable is set to zero, infinity, or some other key value. Make sure that your general expression actually displays the expected behavior! You need to be very careful with limits; sometimes you may run into a paradox if you erroneously convince yourself that an expression must behave a certain way in a certain limit. It is always worthwhile to resolve these paradoxes, as this will help you build up your physical intuition. As with dimensional analysis, careful checking of limits can tell you that an expression is wrong, but it cannot prove that an expression is correct. In this section we focus on using limits wisely, but do not forget to also check units as you did in the previous section.

Here is a simple example that can be used to help remember trigonometric multiple angle formulae.

21. Which of the following expressions might be the correct identity for the sine of a sum of angles?

A) $\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2$
B) $\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2$
C) $\sin(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2$
D) $\sin(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$

Answer: All of these expression are dimensionally correct, so we need to use knowledge of limits. The best limits to start with are ones that simplify an expression, ideally to a level where you have some intuition. Let's start with the limit $\theta_1 = 0$. In that case, $\sin\theta_1 = 0$ and $\cos\theta_1 = 1$. So
A) reduces to $\sin\theta_2 = -\sin\theta_2$ (WRONG)
B) reduces to $\sin\theta_2 = \sin\theta_2$ (COULD BE OK)
C) reduces to $\sin\theta_2 = \cos\theta_2$ (WRONG)
D) reduces to $\sin\theta_2 = \cos\theta_2$ (WRONG)
Note that these simplified equations are wrong if they fail for any value of θ_2 - so C) and D) are wrong because they fail for $\theta_2 = 0$ while A) is wrong because it fails for $\theta_2 \neq 0$. It is useful to realize that you can obtain the formula for the sine of the difference of two angles by letting $\theta_2 \to -\theta_2$ in the correct answer B).

Here is another example from basic physics: rule out the answers from problem 1 (repeated immediately below) using limit arguments.

22. A daredevil is shot out of a cannon at speed v and angle θ from horizontal. Earth's gravitational acceleration, g, is assumed constant, and air resistance is neglected. How far downrange does the daredevil fly?

A) $D = 2v^2 \cos\theta$
B) $D = (2v^2/g) \sin\theta \cos\theta$
C) $D = 2g \sin\theta \cos\theta$
D) $D = 2vg(\cos\theta - \sin\theta)$
E) $D = 2v^2 \sin g$

Answer: First, consider the expected result in various limiting angles. If the cannon is pointed straight upward ($\theta = 90^\circ$), no horizontal progress is made, so $D = 0$. An answer that predicts otherwise is wrong, so (D) is eliminated. Also, although (E) is meaningless because a dimensional quantity is the argument of a sine, it can also be eliminated by limits because it predicts no dependence on angle. When the cannon is pointed horizontally ($\theta = 0$), the daredevil will fall immediately so again $D = 0$. Answer (A) therefore is incorrect. Now let's think of what happens when the velocity v is changed. When $v = 0$, we must have $D = 0$. Answer (C), though, has no v dependence, so it can be eliminated. We find, we did by using units, that only B) can be correct.

23. Which of the following expressions is the correct formula for the cosine of a sum of angles?

A) $\cos(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2$
B) $\cos(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2$
C) $\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2$
D) $\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$

24. Two bodies of masses m_1 and m_2 are placed a distance r apart. What is the strength of the gravitational forces that the bodies exert on each other?

A) $F = G(m_1 + m_2)m_2/r^2$
B) $F = G(m_1 + r)(m_2 + r)/r^2$
C) $F = Gr^2 m_1 m_2$
D) $F = G/(r^2 m_1 m_2)$
E) $F = Gm_1 m_2/r^2$

25. A pendulum bob of mass m is attached to a fixed point by a string of length l. At its lowest point, the bob is a distance h above the floor. What is the frequency of oscillation of the pendulum, if it is in a constant gravitational field of acceleration g?

A) $\omega^2 = g/h$
B) $\omega^2 = gh^2/l$
C) $\omega^2 = g/(l + h)$
D) $\omega^2 = g/l$
E) $\omega^2 = gl$

26. What is the oscillation frequency of an iceberg bobbing up and down in the ocean? Assume that the iceberg is box-shaped with cross sectional area A and g is the constant acceleration of gravity. In equilibrium, the iceberg floats with height H showing above water and height L hidden under water.

A) $\omega^2 = g/L$
B) $\omega^2 = g/H$
C) $\omega^2 = g/(L + H)$
D) $\omega^2 = gH/A^2$
E) $\omega^2 = gA/(HL)$
F) $\omega^2 = gH/[L(H + L)]$

27. A photon is placed in the center of a spherical gas cloud of radius R. If the average distance between scatterings is $l \ll R$, what is the approximate distance d traveled by the photon before it escapes? Here the "$\propto$" sign indicates that the expression is correct down to a dimensionless constant.

A) $d \propto l$
B) $d \propto R^2$

C) $d \propto R^2/l$
D) $d \propto R/l^2$
E) $d \propto l^2/R$

28. A source is emitting sound of speed c_s. You move with speed $v < c_s$, where $v > 0$ means away from the source and $v < 0$ means towards the source. If the sound has frequency f_s as measured in the rest frame of the source, what is the received frequency f_r that you measure?

A) $f_r = \sqrt{(c-v)/(c+v)} f_s$
B) $f_r = \sqrt{(v-c)/(v+c)} f_s$
C) $f_r = \sqrt{(c+v)/(c-v)} f_s$
D) $f_r = \sqrt{(v+c)/(v-c)} f_s$

29. Suppose that a block of mass m starts from rest and slides down an inclined plane of height h, which has a slope angle θ (where $\theta = 0$ means a horizontal line and $\theta = \pi/2$ means a vertical drop). The inclined plane has a curved tip so that the downward component of velocity is smoothly transformed into a horizontal component. The gravitational acceleration is a constant value g, and we neglect friction with the plane and air resistance. What is the speed of the block when it reaches the bottom of the inclined plane?

A) $v = \sqrt{gh} \cos\theta$
B) $v = m\sqrt{h} \sin\theta$
C) $v = \sqrt{2gh}$
D) $v = \sqrt{Gm/h}$

30. Suppose that a sphere with moment of inertia I and radius r rolls without slipping down an inclined plane of height h and slope angle θ. Again, the gravitational acceleration is a constant value g and we neglect friction and air resistance. What is the linear speed of the sphere when it reaches the bottom?

A) $v = \sqrt{2gh + Ir^2}$
B) $v = \sqrt{2gh/(1 + I/mr^2)}$
C) $v = \left(1 + I/mr^2\right) \sqrt{2gh \tan\theta}$
D) $v = \sqrt{2gh(1 + I/mr^2)}$

31. Let a particle orbit in a circle a distance h above a the surface of a planet of mass M and radius R (the particle mass is assumed very small). What is the angular momentum per unit mass of the particle?

A) $\ell = \sqrt{GM(R+h)}$
B) $\ell = \sqrt{GM}(Rh)^{1/4}$
C) $\ell = \sqrt{GMR^2/h}$
D) $\ell = \sqrt{GMh^2/R}$

32. A gas of particles with mass m and temperature T (and hence thermal energy kT) is placed in a tube with cross-sectional area A in a constant gravitational field of acceleration g. If the pressure at the bottom of the tube is P_0, what is the pressure at a height h above the bottom?

A) $P = kT/(Ah)$
B) $P = P_0(mgh/kT)$
C) $P = P_0 \exp(mgh/kT)$
D) $P = P_0 \exp(-kT/mgh)$
E) $P = P_0 \exp(-mgh/kT)$

33. On a space station with no gravity, a BB is shot into a room full of billiard balls. The BB has velocity v, the billiard balls have diameter d, and the number density of billiard balls is n. What is the mean free path ℓ traveled by the BB before it hits a ball?

A) $\ell \propto nd^2$
B) $\ell \propto 1/(nd^2)$
C) $\ell \propto (v/c)/(nd^2)$

34. A telescope with aperture D observes a source at a wavelength λ. Diffraction limits the angular resolution θ. What is that limit?

A) $\theta = 1.22\lambda D$
B) $\theta = 1.22D/\lambda$
C) $\theta = 1.22\lambda/D$
D) $\theta = 1.22/(\lambda D)$

35. You draw an n-sided polygon on a sphere of radius R. The sum of the interior angles of the polygon is θ radians. What is the area S of the polygon, as measured on the surface of the sphere?

A) $S = \theta R^2$
B) $S = [\theta - (n-2)\pi] R^2$
C) $S = \frac{1}{2} [\theta - (n+2)\pi] R^2$
D) $S = \frac{1}{2} [\theta + (n+2)\pi] R^2$

36. What is the area of a regular polygon of n sides, each with length x?

A) $A = nx^2 [\cos(\pi/n)/\sin(\pi/n)]$
B) $A = \frac{1}{4}nx^2 [\cos(\pi/n)/\sin(\pi/n)]$
C) $A = nx^2 [\cos(n\pi)/\sin(n\pi)]$
D) $A = \frac{1}{4}nx^2 [\sin(\pi/n)/\cos(\pi/n)]$

37. A light ray passes by a spherical star of mass M and radius R, with a trajectory that has an impact parameter $b > R$. In radians, what is the total angular deflection $\Delta\theta$ of the ray due to the gravity of the star, assuming that $\Delta\theta \ll 1$?

A) $\Delta\theta = 4GM/bc^2$
B) $\Delta\theta = 4bc^2/GM$
C) $\Delta\theta = 4R/b$
D) $\Delta\theta = 4GM/Rc^2$

38. A narrow hole is drilled through the center of a sphere of mass M and radius R. A particle of mass $m \ll M$ is dropped from rest at an entrance of the hole, and will therefore fall all the way through, stop at the other entrance, and come back. What could the oscillation frequency be?

A) $\omega = \sqrt{GM/R^3}$
B) $\omega = \sqrt{Gm/R^3}$
C) $\omega = \sqrt{GM^2/mR^3}$
D) $\omega = \sqrt{c^4/GMR}$

39. A bouncing ball starts at height h. In each bounce it loses a fraction ϵ of its energy. What is the total distance it travels before coming to rest?

A) $d = h(1-\epsilon)/\epsilon$
B) $d = h\epsilon/2$
C) $d = h(2-\epsilon)/\epsilon$
D) $d = h\epsilon(1+\epsilon)$

40. Consider a planet of mass m orbiting in a circle of radius R around a star of mass $M \gg m$. The "Hill sphere" is the volume inside of which the gravity of the planet is more important than the tidal force exerted by the gravity of the star. What is the radius of the Hill sphere?

A) $r_H = (m/3M)^{1/3}R$
B) $r_H = (M/m)^{1/3}R$
C) $r_H = R$
D) $r_H = (m+M)^{1/3}R$

41. One idea for sending spacecraft to other stars is by radiative acceleration. A laser beam of luminosity L is aimed at such a spacecraft. The beam has a solid angle $\Delta\Omega$. The spacecraft has mass m and a perfect mirror of area A pointed towards the laser. At the moment when the spacecraft is a distance r from the laser (assuming $r^2\Delta\Omega > A$), what is the acceleration produced? Assume the craft is moving non-relativistically.

A) $a = L\Delta\Omega r^2/(mcA)$
B) $a = mc^4/L$
C) $a = LA\Delta\Omega/(mcr^2)$
D) $a = LA/(mcr^2\Delta\Omega)$

42. A sphere of radius R has a cylindrical hole cut through it. The distance from the top lip of the hole to the bottom lip of the hole is $2d$. What is the volume of the sphere that remains?

A) $V = \frac{4}{3}\pi d^3$
B) $V = \frac{4}{3}\pi R^3$
C) $V = \frac{4}{3}\pi R^4/d$
D) $V = \frac{4}{3}\pi(d+R)^2R$

4. TAKING ADVANTAGE OF SYMMETRIES

Symmetries are fundamental in physics (and astronomy!). Problems can have symmetry about a point (spherical symmetry), a line (cylindrical or axial symmetry), or a plane (mirror symmetry). You can use symmetries in two ways: 1) to check your final answer to a problem or, with a little more effort, 2) to simplify the derivation of that final answer. As an example, time-independent central forces (like gravity) have spherical symmetry because the force depends only on the distance from the origin. In this case, spherical symmetry means that once we find one solution (e.g. a particular ellipse for gravity), all other possible orientations of this solution in space are also solutions.

Another type of symmetry could be called a symmetry of labeling. In many problems, it is clear that simply renaming two identical things can't change anything fundamental about the system. For example, consider two objects of mass m_1 and m_2 moving in circular orbits around each other, bound by gravity, separated by a distance a. What is the frequency of rotation? A guess like $\omega = \sqrt{G(2m_1+m_2)/a^3}$ can't be right, because the answer would change simply by switching the labels on the masses.

43. There are two foci in an ellipse. As the ellipse is slowly changed into a circle (a special case of an ellipse), what happens to the foci?

A) They move toward the ends of the ellipse major axis and end up along a diameter of the circle.
B) They merge and annihilate. A circle has no focus.
C) They merge into a single focus at the center.

44. The two foci of an ellipse are points in its interior whose location depends only on the geometry of the ellipse. We now define "bogi" in an arbitrary, but analogous way (i.e. the location of the points depend only on the ellipse geometry). If there are a finite number of bogi, which of the following properties must hold?

A) There are at least two bogi
B) There is only a single bogus
C) There are an even number of bogi
D) All the bogi must lie on the ellipse major or minor axes
E) The bogi must be inside the ellipse
F) The number of bogi not on the major axis is even
G) The number of bogi not on the minor axis is even

45. An object travels along an elliptical orbit under the influence of an attractive time-independent central force. If the long axis of the ellipse is oriented along the x-axis and the speed of the object satisfies v(x,y) = v(x,-y), what can be said about the location of the force center?

A) It is outside the ellipse
B) It is at the center of the ellipse
C) It is along the minor axis of the ellipse
D) It is along the major axis of the ellipse
E) It is a focus of the ellipse

46. In two dimensions, the orbit of an assumed massless moon around a planet of mass M is elliptical and can be defined in terms of four orbital elements. These elements are a, the semimajor axis of the ellipse, e, its eccentricity, ω, an angle defining the ellipse orientation, and ν, the angular position of the moon along its elliptical orbit. The transformation $e \to -e, \omega \to \omega + \pi, \nu \to \nu + \pi$, leaves the elliptical orbit unaltered. What formula defines the specific angular momentum h of the orbit?

A) $h^2 = GMa(1-e)$
B) $h^2 = GMa(1-e^2)$
C) $h^2 = GMa(1-e^3)$
D) $h^2 = GMa(1-e\cos\omega)$
E) $h^2 = GMa(1+e^2)$

47. A star of average radius R is rotating with angular frequency ω. We define the sign of ω such that if $\omega > 0$ then the star is rotating west to east like the Earth, whereas if $\omega < 0$ then the star is rotating east to west. Rotation will distort the radius of the star. To lowest order in ω, what will be the deviation of the equatorial radius from R?

A) $\Delta r \propto \omega$
B) $\Delta r \propto \omega^2$
C) $\Delta r \propto \omega^3$

48. In general relativity, a rotating object drags spacetime with it. Suppose one has a particle in a circular equatorial orbit around a black hole of mass M spinning clockwise at a small frequency ω. Define $\Delta\Omega$ as the difference in orbital frequencies measured at infinity between a clockwise and a counterclockwise orbit, both at radius r. How does $\Delta\Omega$ depend on ω?

A) $\Delta\Omega \propto \omega^0$
B) $\Delta\Omega \propto \omega^1$
C) $\Delta\Omega \propto \omega^2$

49. You have coffee in a circular cup. The coffee is initially at rest, at height h_0. You stir it so that the coffee acquires a uniform angular velocity ω (here $\omega > 0$ if the motion is counterclockwise as seen from above, and $\omega < 0$ if the motion is clockwise). The coffee rises at the sides of the cup as a result, by an amount Δh. Which of the following could be true?

A) $\Delta h \propto \omega$
B) $\Delta h \propto \omega^2$
C) $\Delta h \propto \omega^3$

50. A white dwarf of mass M has reached a stable equilibrium radius R. Its total energy is therefore a minimum. If the white dwarf's radius is changed by ΔR ($\Delta R < 0$ means shrinking the star; $\Delta R > 0$ means expanding it) then which of the following could be true about the change ΔE in the total energy?

A) $\Delta E \propto (\Delta R)^0$
B) $\Delta E \propto (\Delta R)^1$
C) $\Delta E \propto (\Delta R)^2$
D) $\Delta E \propto (\Delta R)^3$

51. An nth order algebraic equation is written as $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 = 0$. This can be factored to read $(x-r_1)(x-r_2)\cdots(x-r_n) = 0$, so there are n roots $r_1, r_2, \cdots r_n$. Count multiple roots separately (e.g., if $r_i = r_j$, count them as two roots). If all the roots are real and a_0 is positive, which of the following is guaranteed to be true?

A) The number of zero roots is odd.
B) The number of negative roots is odd.
C) The number of negative roots is even.
D) The number of positive roots is even.
E) The number of positive roots is odd.

52. Which of the following could be the correct expression for the dimensionless amplitude of a gravitational wave, a ripple in spacetime due to moving masses, as received from a circular binary (stars of mass M and m) at a distance r with a binary frequency $f_{\rm bin}$?

A) $h = 2^{5/3}(4\pi)^{1/3}(G^{5/3}/c^4)f_{\rm bin}^{2/3}M^2(M+m)^{-1/3}(1/r)$
B) $h = 2^{5/3}(4\pi)^{1/3}(G^{5/3}/c^4)f_{\rm bin}^{2/3}Mm(M-m)^{-1/3}(1/r)$
C) $h = 2^{5/3}(4\pi)^{1/3}(G^{5/3}/c^4)f_{\rm bin}^{2/3}Mm(M+m)^{-1/3}(1/r)$
D) $h = 2^{5/3}(4\pi)^{1/3}(G^{5/3}/c^4)f_{\rm bin}^{2/3}m(M+m)^{-1/3}(1/r)$

53. A collection of particles is initially distributed in a spherical shell. It is set rotating at an angular frequency ω. The new location of the particles depends only on ω and the cylindrical distance from the rotational axis. Which of the following is a possible location for the particles that are farthest away from the original center of the sphere?

A) 180° west longitude
B) 45° north latitude
C) 45° south latitude

D) The rotational equator

54. Consider an odd function of one variable, $f(x) = -f(-x)$. What can one say about its value at $x = 0$?

A) $f(0)$ is positive
B) $f(0)$ is negative
C) $f(0) = 0$
D) It is impossible to say

55. How many roots of an odd function $f(x)$ can there be in the interval from $x = -a$ to $x = a$?

A) The number of roots is even
B) The number of roots is odd
C) The number of roots could be even or odd

56. How many roots of an even function $g(x) = g(-x)$ can there be in the interval from $x = -a$ to $x = a$?

A) The number of roots is even
B) The number of roots is odd
C) The number of roots could be even or odd

57. Let the parity of a function be $+1$ if the function is even, -1 if the function is odd. Consider a function that is the product of n even functions and m odd functions. What is the parity of this product function?

A) $(-1)^n$
B) $(-1)^m$
C) $(-1)^{n+m}$
D) $(-1)^{n-m}$
E) $(-1)^{m-n}$

58. An object is rotated θ_1 degrees around axis 1. It is then rotated θ_2 degrees around axis 2 (which may be the same as axis 1 or it may be different). Call this final orientation 1. Starting from the original orientation, the same rotations are performed but in the opposite order. Call this new final orientation 2. Compare orientation 1 with orientation 2.

A) They are always the same
B) They are always different
C) They are sometimes the same

5. CONSERVATION OF MOMENTUM, ANGULAR MOMENTUM, AND ENERGY

Conservation laws are fundamental in physics, often allowing otherwise intractable problems to be solved with ease. It is important to be able to look at a problem and immediately recognize which conservation law or laws give the most insight into a problem. Energy is conserved in an isolated system, but is often transferred to heat in dissipative systems. A system's momentum is conserved if no external forces act on it and its angular momentum is conserved if there are no external torques. All of the problems in this section can be solved by using one or more conservation laws.

59. A particle moves under the influence of a central force. That is, the force points towards or away from a fixed central point, and the magnitude of the force depends only on the distance from the central point. Which of the following quantities of the particle must be conserved?

A) Kinetic energy
B) Linear momentum
C) Angular momentum

60. A particle of mass m_1 moving at velocity v_1 hits another particle of mass m_2 at rest. After the elastic collision, what is the maximum angle between the initial velocity of particle 1 and the new velocity of particle 2?

A) 45 degrees
B) 60 degrees
C) 90 degrees
D) 120 degrees
E) 180 degrees

61. Consider an inclined plane of length l and angle 45° from horizontal. A sphere of mass m and a block, also of mass m, are both placed at the top. The sphere rolls without sliding; the block slides frictionlessly without rolling. Ignoring air resistance in both cases, which reaches the bottom first?

A) The sphere reaches the bottom first.
B) The block reaches the bottom first.
C) They reach the bottom at the same time.

62. A solid marble of mass m and radius r rolls without slipping down a ramp toward a loop of radius R. How high must the ramp be in order that the marble just avoid falling off the top of the loop? Assume $r \ll R$.

A) $h = R$
B) $h = 4R(1 - r/R)$
C) $h = 2.7R$
D) $h = 2R$
E) $h = 4r(1 + r/R)$

63. If the marble in the above problem is hollow, how is the minimum height affected?

A) It does not change
B) It is higher
C) It is lower

64. A ball is thrown straight upwards into still air. It travels a height h, then falls back and is caught at the same height that it was released. Compare the time to go up with the time to come down taking air resistance into account.

A) It takes longer to go up than to come down.
B) It takes the same time to go up as to come down.
C) It takes longer to come down than to go up.

65. In general relativity, a planet orbiting around a star emits energy in the form of gravitational radiation. Assuming that the planet is always in a nearly circular orbit, and ignoring other effects, what happens to the orbital speed?

A) The orbital speed decreases with time.
B) The orbital speed stays the same.
C) The orbital speed increases with time.

66. Globular clusters are relatively compact collections of $10^5 - 10^6$ stars within a few parsecs of each other. Gravitational interactions between the stars conserve the overall energy of the cluster. If the cluster is considered in isolation and no stars escape, which of the following could happen to the cluster over time?

A) The cluster expands as a whole.
B) The cluster contracts as a whole.
C) The outer portion of the cluster expands and the inner portion contracts.

67. Two stars are placed in orbit around each other. The initial orbit is highly eccentric, with semimajor axis a. On their first close pass, tidal coupling between the stars causes them to oscillate and heat up. Compare the new semimajor axis a' to a.

A) $a' < a$
B) $a' = a$
C) $a' > a$

68. A photon of wavelength λ is emitted in a gravitational field. Assume it is outside a nonrotating spherical object of mass M, and is initially a distance r from the center of the object. It is received at a radius r' from the center of the object. Assuming weak gravity, what is the new wavelength λ' of the photon as measured at the receiver?

A) $\lambda'/\lambda = 1 + (GM/rc^2 - GM/r'c^2)$
B) $\lambda'/\lambda = 1 + (GM/r'c^2 - GM/rc^2)$
C) $\lambda' = \lambda$

69. A thin disk is placed in an isotropic radiation field. One side of the disk is a perfect reflector with outward normal $\hat{z}$ and the other is a perfect absorber with outward normal $-\hat{z}$. If the disk is initially at rest and nonrotating, what happens when it is placed in the radiation field?

A) It remains at rest and nonrotating.
B) It rotates but remains at rest.
C) It rotates and accelerates in the $-\hat{z}$ direction.
D) It rotates and accelerates in the $\hat{z}$ direction.
E) It does not rotate but accelerates in the $-\hat{z}$ direction.
F) It does not rotate but accelerates in the $\hat{z}$ direction.

70. A spacecraft flies near Jupiter and uses the gravitational slingshot effect to increase its energy, thereby speeding its journey to Saturn. What happens to Jupiter as a result of the interaction?

A) Jupiter loses a little energy and its orbit contracts slightly.
B) Jupiter is unaffected by the spacecraft's maneuver.
C) Jupiter gains a little energy and its orbit expands slightly.

71. A planet is in a circular orbit around a star. The star is nonrotating, and loses mass via a radial wind. Assuming that the planet does not interact with the wind directly, and ignoring other effects, what happens to the orbital radius of the planet?

A) The orbital radius decreases with time.
B) The orbital radius remains constant.
C) The orbital radius increases with time.

72. The Moon is outside synchronous orbit for the Earth, meaning that the Moon's orbital period is longer than the Earth's rotational period. Tides raised on Earth by the Moon push the Moon away from the Earth by a few centimeters per year. What does this do to Earth's rotation rate?

A) The rotation rate decreases with time.
B) The rotation rate remains constant.
C) The rotation rate increases with time.

73. Phobos, the larger of Mars' two moons, is inside synchronous orbit; that is, its orbital period is shorter than Mars' rotation period. Tides raised on Mars by Phobos therefore tend to increase Mars' rotation rate. What does this do to the orbital radius of Phobos?

A) The orbital radius decreases with time.
B) The orbital radius remains constant.
C) The orbital radius increases with time.

74. Pluto and Charon are in mutually synchronous orbit: their rotation periods are equal to their orbital periods around each other. Considering the Pluto-Charon system as isolated, what happens to the orbital radius?

A) The orbital radius decreases with time.
B) The orbital radius remains constant.

C) The orbital radius increases with time.

6. SIMPLE ESTIMATION & INTUITION

At the end of a derivation you sometimes want a numerical answer instead of just an equation. If you just plug and chug on a calculator, the answer you come up with may seem authoritative but could be drastically off. For instance, by mistake you might divide by a small number rather than multiplying by it. You can catch simple errors like this by applying a certain amount of common sense. A feel for what the approximate answer should be will help guide you. For example:

75. The mass of the coffee in a standard mug is about:

A) 0.003 kg
B) 0.3 kg
C) 30 kg
D) 3000 kg

Answer: the volume of the mug is not specified, so we cannot get an exact numerical answer. However, answer (D) can obviously be eliminated, because 3000 kg=3 tons which would make it impossible to pick up a cup of coffee! Answer (C) is also too large; a mug weighing 30kg$\approx$ 65lbs would be difficult to manage, especially with one hand! Answer A) is too light 0.003 kg is only about a tenth of an ounce - a tiny fraction of the mass of the mug itself. Therefore, (B) is closest to the correct answer. Often it is valuable to do a "quick and dirty" estimation to see if the answer matches roughly with what you get from your calculator - if it doesn't you should be suspicious!

76. What is the maximum speed of a commercial jet?

A) 10 km/hr
B) 30 km/hr
C) 100 km/hr
D) 300 km/hr
E) 1000 km/hr
F) 3000 km/hr

77. Approximately how fast can a kangaroo move?

A) 10^{-2} m s^{-1}
B) 10^{-1} m s^{-1}
C) 10^{0} m s^{-1}
D) 10^{1} m s^{-1}
E) 10^{2} m s^{-1}

78. At a typical walking speed with no breaks, about how long would it take to travel a distance equal to the circumference of the Earth?

A) A day
B) A week
C) A month
D) A year

79. What is the approximate average power usage per person in the United States?

A) 1 Watt
B) 10^4 Watts
C) 10^8 Watts
D) 10^{12} Watts

80. Suppose you have a perfectly absorbing surface oriented towards the Sun at a distance equal to the Earth-Sun distance. What will be the approximate blackbody temperature of the surface?

A) 4 K
B) 40 K
C) 400 K
D) 4000 K

81. What is the approximate world record time for the running of a 100 mile ultramarathon?

A) Half an hour
B) Half a day
C) Half a week
D) Half a year

82. The avian record-holder for speed in level flight is the spine-tailed swift. What is its top speed?

A) 5 m/s
B) 50 m/s
C) 500 m/s
D) 5000 m/s

83. Roughly what is the human world record for swimming 100 meters?

A) 1 second
B) 1 minute
C) 1 hour
D) 10 hours

84. About how long would it take an anvil to drop 5 meters from rest?

A) 0.1 second
B) 1 second
C) 10 seconds
D) 1 minute

85. Roughly what is the record for distance in throwing a baseball?

A) 1.3 meters
B) 13 meters
C) 130 meters
D) 1300 meters

86. What is an approximate size range for stars on the main sequence, such as the Sun?

A) $10^1 - 10^3$ km
B) $10^5 - 10^7$ km
C) $10^9 - 10^{11}$ km

87. What is the approximate surface temperature of the Sun?

A) 6 K
B) 60 K
C) 600 K
D) 6,000 K

88. What is the best estimate for the world record in the bench press?

A) 30 kg
B) 300 kg
C) 3,000 kg
D) 30,000 kg

7. ESTIMATION

Some quantities are governed by simple formulae which can be used to provide a quick approximation. The idea here is to estimate rather than calculate - imagine that you are given these problems on a test and have no calculator handy. You can catch many common mistakes in a long calculation by occasionally estimating how large a result you expect. In fact, with some practice, you can often approximate the answer to a given problem faster by hand that with a calculator. For example:

89. How long is a billion seconds?

A) 3 hours
B) 3 days
C) 3 months
D) 3 years
E) 30 years
F) 300 years

Answer: A good place to start is with the number of seconds in a year. We can estimate this by remembering that there are 365 days in a year, 24 hours in a day, 60 minutes in an hour and 60 seconds in a minute. So $T = 365{\times}24{\times}60{\times}60$ seconds. We don't need a very accurate value given the factor of ten difference in the answers above, so we can estimate: 365×24 is about 10,000, while 60×60 is 3600. So there are about 3.5×10^7 seconds in a year. A billion seconds is about $10^9/3.5 \times 10^7 \approx 30$ - answer E) is the correct choice.

This next example is slightly more involved, requiring a few successive steps.

90. Approximately how fast does the Earth move in its orbit?

A) 10^0 m s^{-1}
B) 10^2 m s^{-1}
C) 10^4 m s^{-1}
D) 10^6 m s^{-1}

Answer: You might have no idea of how fast the Earth is moving, but you probably know that the Earth is about 93,000,000 miles from the Sun. The Earth goes around a nearly circular orbit of length 2π times this in one year. So, making use of the formula $v = 2\pi r/T$, we can relate the velocity v to the orbital radius r and the orbital period T. Now the four answers that we are considering differ by factors of 100, so we don't have to do these calculations exactly. We estimate. Since the answers are given in terms of meters and seconds we need to translate r and T into these units. Remembering that a mile is about 1.6 kilometers we have: r=93,000,000 miles $\approx 150,000,000$km $\approx 1.5 \times 10^{11}$m. From the previous problem, there are about 3.5×10^7 seconds in a year. Finally we have $v = 2\pi r/T \approx 2\pi(1.5 \times 10^{11}/3.5 \times 10^7) \approx 10^{12}/3.5 \times 10^7 \approx 3 \times 10^4$. The correct answer is about $3{\times}10^4$ m/s, and answer C) is closest.

91. Approximately how high is a stack of a million sheets of paper?

A) 0.01 m
B) 1 m
C) 100 m
D) 10^4 m

92. How far does light travel in one year?

A) 10^{13}m
B) 10^{14}m
C) 10^{15}m
D) 10^{16}m
E) 10^{17}m

93. The radius of the icy former planet Pluto is about 1150km. What is the approximate mass of Pluto?

A) 10^{20}kg
B) 10^{22}kg

C) 10^{24}kg
D) 10^{26}kg
E) 10^{28}kg

94. The radius of the Earth is about 6400km. What is the approximate mass of the Earth?

A) 6×10^{21}kg
B) 6×10^{24}kg
C) 6×10^{27}kg
D) 6×10^{30}kg
E) 6×10^{33}kg

95. A 20 Mton nuclear bomb explosion releases 8×10^{16} Joules of energy. How much matter needs to be converted to energy to power the explosion?

A) 10^{-3} kg
B) 1 kg
C) 1 ton
D) 1000 tons
E) 10^{6} tons

96. If the radius of the Earth is 6400 km and the average depth of the ocean is 1 km, what fraction of the Earth's total mass do the oceans account for?

A) 10^{-7}
B) 10^{-6}
C) 10^{-5}
D) 10^{-4}
E) 10^{-3}

97. The most massive living tree is the General Sherman sequoia. This tree is 83 meters high and 11 meters in diameter at its base. Approximately what is its mass?

A) 200 kg
B) 20 tons
C) 2000 tons
D) 2 million tons

98. About how many molecules are in a raindrop?

A) 10^{15}
B) 10^{21}
C) 10^{27}
D) 10^{29}

99. If you were to lay all the atoms of your body in a row, roughly how long a line would be formed?

A) 5 meters
B) 5 km
C) 5 astronomical units
D) 5 light years

100. About how many letters, spaces, and other symbols are in an average novel?

A) 10^{3}
B) 10^{6}
C) 10^{9}

101. Given the above, roughly how many novels could a good typist produce per year at 500 characters per minute, 8 hours per day, assuming no pauses and no mistakes?

A) 10
B) 100
C) 1,000
D) 10,000

102. How large an area would a 100 meter deep lake have to have to contain as much mass in water as the combined mass of all the people on Earth?

A) 3 m^2
B) 3 km^2
C) 3,000 km^2
D) 3,000,000 km^2

103. The largest office building in the world is the Pentagon. What is its approximate total floor area?

A) 6,000 m^2
B) 600,000 m^2
C) 6×10^{7} m^2
D) 6×10^{9} m^2

104. What is the approximate combined mass of all the food and drink you have ever consumed?

A) $1 - 3 \times 10^{2}$ kg
B) $1 - 3 \times 10^{4}$ kg
C) $1 - 3 \times 10^{6}$ kg
D) $1 - 3 \times 10^{8}$ kg

8. INTERMEDIATE PROBLEMS

These problems may require checking of units, limits, or symmetry to rule out some of the answers.

105. The energy density of a magnetic field is proportional to $B^2/(2\mu_0)$. Therefore, the magnetic pressure is proportional to:

A) $P \propto B/(2\mu_0)$
B) $P \propto B^2/(2\mu_0)$
C) $P \propto B^3/(2\mu_0)$
D) $P \propto B^4/(2\mu_0)$
Here the proportionality sign indicates a dimensionless factor.

106. The electrostatic energy of two charges e separated by a distance r is $e^2/(4\pi\epsilon_0 r)$. Which of the following could be expressions for angular momentum, if v is a velocity?

A) $L = e^2/(4\pi\epsilon_0)$
B) $L = e^2/(4\pi\epsilon_0 v)$
C) $L = em$
D) $L = v^2/er^2$

107. A particle with electric charge e is accelerated at a rate a. What is the power radiated by the charge as a result? As usual, c is the speed of light.

A) $P \propto e/a$
B) $P \propto 4\pi\epsilon_0 a^2/e^2$
C) $P \propto e^2 a^2/(4\pi\epsilon_0 c^3)$
D) $P \propto eac^2$
E) $P \propto \cos(ea)$

108. The fine structure constant α_f is a dimensionless quantity important in a number of quantum electrodynamic calculations. Which of the following could be the fine structure constant?

A) $\alpha_f = e^2/(4\pi\epsilon_0 \hbar c)$
B) $\alpha_f = \hbar^2/(4\pi\epsilon_0 ec)$
C) $\alpha_f = c^2/(4\pi\epsilon_0 e\hbar)$
D) $\alpha_f = e^2/(4\pi\epsilon_0 \hbar^2 c^2)$

109. The plasma frequency ω_e is a frequency such that waves in a plasma with less than this frequency are attenuated rapidly, whereas waves with greater than this frequency can propagate with comparatively little loss. Which of the following could be the plasma frequency? Here m_e is the mass of the electron and N_e is the number density (number per m^3) of free electrons.

A) $\omega_e = N_e e^2/(\epsilon_0 m)$
B) $\omega_e = \sqrt{N_e e^2/(\epsilon_0 m)}$
C) $\omega_e = \epsilon_0 m/(N_e e^2)$
D) $\omega_e = \sqrt{\epsilon_0 m/(N_e e^2)}$

110. A collection of gas in space, initially uniform, can be unstable to gravitational perturbations. For low gas masses, thermal motions spread the gas out faster than gravity can collect it, but for large masses gravity wins and amplifies any initial density enhancement. The higher the temperature, the higher the mass required; the higher the density, the less mass required. Suppose the sound speed is c_s and the density is ρ. Which of the following could be the Jeans mass, which is the lowest mass that is unstable?

A) $M_J = (\pi^{5/2}/6)\rho c_s^2(G\rho)$
B) $M_J = (\pi^{5/2}/6)\rho c_s^3(G\rho)^{3/2}$
C) $M_J = (\pi^{5/2}/6)\rho c_s^3(G\rho)^{-3/2}$
D) $M_J = (\pi^{5/2}/6)\rho^3/G$

111. What is the difference between the gravitational acceleration at Earth's surface and the acceleration a height h above the surface? Treat the Earth as a perfect sphere of radius R and mass M.

A) GMh/R^3
B) $GM(1/R^2 - 1/(R+h)^2)$
C) $(GM/(R(R+h))$
D) $GM(1/R^2 - 1/(R(R+h)))$
E) $GM/(R+h)^2$

112. Skyhook, Inc. has set up shop around a spherical planet of mass M, radius R, and angular velocity ω radians per second. They put a satellite in synchronous orbit above the equator, at a radius $R_{\rm syn}$ such that the orbital angular velocity equals ω. They want to have a cable dangling from the satellite to the surface, to facilitate transport of materials from the surface to the satellite. If the mass per unit length of the cable is μ, what is the effective force that must be exerted on the cable to support it?

A) $F = GM\mu/R$
B) $F = GM\mu\left[1/R - 3/(2R_{\rm syn}) + R^2/(2R_{\rm syn}^3)\right]$
C) $F = GM\mu(R_{\rm syn}/R^2 - 5/(2R))$
D) $F = GM\mu\left[1/R + 3/(2R_{\rm syn}) + R^2/(2R_{\rm syn}^3)\right]$

113. A block of mass m_2 slides without friction down a triangular wedge of mass m_1 which rests on a frictionless table. If m_2 is initially a height h above the table, and the slope of the triangular block is determined by the angle $\theta < 90^\circ$ (where $\theta = 0$ means horizontal), what is the speed of recoil of the triangular wedge?

A) $v^2 = gh$
B) $v^2 = gh\tan\theta$
C) $v^2 = ghm_2\tan\theta/(m_1+m_2)$
D) $v^2 = ghm_2/(m_1+m_2)$
E) $v^2 = ghm_2/m_1$

114. Two stars of masses m_1 and m_2 move along an elliptical orbit with semimajor axis a and eccentricity e. What is the angular frequency ω of orbit, where $w = 2\pi/P$ and P is the orbital period?

A) $\omega^2 = G(m_1+m_2)/a^3$
B) $\omega^2 = Gm_1/a^3$
C) $\omega^2 = G(m_1+m_2)e/a^3$

D) $\omega^2 = Gm_1m_2/((m_1+m_2)a^3)$
E) $\omega^2 = G(m_1+m_2)/a^4$

115. A planet is in a circular orbit around a star. Its rotational axis is tilted by an angle $\theta_{\text{tilt}} \leq 90^\circ$ relative to the axis of its orbit. Consider a point on the planet, at a polar angle θ (which is 0° at the North Pole and 180° at the South Pole) and azimuthal angle ϕ. Over the course of the year, what is the maximum apparent angle above the horizon reached by the star, as seen from that point? Assume that the planet rotates much faster than it revolves.

A) $\theta_{\text{max}} = \min[90^\circ, \theta + \theta_{\text{tilt}}]$
B) $\theta_{\text{max}} = \min[90^\circ, \theta + \phi]$
C) $\theta_{\text{max}} = \min[90^\circ, \theta_{\text{tilt}} + \phi]$
D) $\theta_{\text{max}} = \min[90^\circ, 90^\circ - |90^\circ - \theta| + \theta_{\text{tilt}}]$

116. A triangle has sides a, b, and c, with an angle C between sides a and b. Which of the following could be an expression for c^2?

A) $c^2 = a^2 - b^2 + 2ab\sin C$
B) $c^2 = a^2 + b^2 - 2ab\cos C$
C) $c^2 = ab\sin C$
D) $c^2 = \sqrt{a^4+b^4} - 2ab\cos C$

117. A mass m attached to the ceiling by a massless cord of length l moves in a horizontal circle of radius r with uniform angular velocity w. If there is a uniform gravitational acceleration g the tension in the cord is:

A) mgr/l
B) $m\sqrt{(w^2r^2+g^2)}$
C) $m\sqrt{(w^4r^2+g^2)}$
D) $m\sqrt{(w^2l^2+g^2)}$
E) $m\sqrt{(w^4l^2+g^2)}$

118. A sphere of radius R moves very slowly, with velocity v, through a fluid with dynamic viscosity η (the units of η are kg m^{-1} s^{-1}). What is the drag force on the sphere?

A) $F = 6\pi\eta Rv$
B) $F = 6\pi\eta R^2/v$
C) $F = 6\pi Rv/\eta$
D) $F = 6\pi R/(\eta v)$

119. A sphere of radius R moves quickly (but slower than the speed of sound), with velocity v, through a fluid of density ρ. What is the drag force on the sphere?

A) $F \propto \rho Rv$
B) $F \propto \rho v/(R^2)$
C) $F \propto \rho R^2v^2$
D) $F \propto \rho vR^2$

120. Which of the following is the best guess for the mass of a skyscraper such as the Empire State Building?

A) 0.1 kg
B) 100 kg
C) 10^5 kg
D) 10^8 kg

121. The gravitational acceleration at the Earth's surface is g. How strong is the gravitational acceleration at the center of the Earth?

A) g
B) $g/2$
C) $g/4$
D) 0
E) $2g$

122. The gravitational acceleration at the Earth's surface is g. How strong is the gravitational acceleration at one Earth radius above the Earth's surface? Assume that the Earth is a uniform sphere.

A) g
B) $g/2$
C) $g/4$
D) 0
E) $2g$

123. The gravitational acceleration at the Earth's surface is g. How strong is the gravitational acceleration halfway to the center of the Earth?

A) g
B) $g/2$
C) $g/4$
D) 0
E) $2g$

124. Each time a ball hits the ground it deforms and bounces back with 90% of its original height. How many bounces does it take to decrease to 1/2 the original height?

A) 1 bounce
B) 3 bounces
C) 6 bounces
D) 10 bounces
E) 20 bounces

125. Two thin hoops hang from nails in the wall. One hoop has a radius four times as large as the other. What is the small-oscillation period of the larger hoop divided by the small-oscillation period of the smaller hoop?

A) 4
B) 2
C) 1
D) 1/2
E) 1/4

126. The total energy E of a particle with rest mass m_0 moving at speed v is most closely given by which of these expressions? Here c is the speed of light and we assume $v \ll c$.

A) $E^2 = m_0^2c^4 + m_0^2c^2v^2$
B) $E^2 = m_0^2v^4 + m_0^2c^2v^2$
C) $E^2 = m_0^2c^4 + m_0^2v^4$
D) $E^2 = m_0^2c^4 + m_0^2c^2v^2/4$

127. A pendulum bob is released from rest when its string is horizontal and travels along a circular arc. What is the magnitude of the total acceleration as a function of the angle θ that the string makes with the horizontal?

A) $g\sin\theta$
B) $g\cos\theta$
C) $g\sqrt{3\cos^2\theta+1}$
D) $g\sqrt{3\sin^2\theta+1}$
E) $2g\sin\theta$
F) $2g\cos\theta$

128. A uniform rod of length L and mass M is balanced on a fulcrum with a mass m suspended from one end of the rod. How far is the fulcrum from the center of mass of the rod?

A) $Lm/(2M+2m^2/M)$
B) $Lm^2/(2M^2+2m^2)$
C) $Lm/(M+m)$
D) $Lm/(2M+2m)$
E) $Lm/(2M-2m)$

129. At a distance h above the surface of the Earth (with radius R), the distance d to the horizon is given by:

A) $d^2 = 2R^2h^2$
B) $d^2 = 2Rh$
C) $d^2 = 2Rh + h^2$
D) $d^2 = 2Rh + 2h^2$
E) $d^2 = R^2 + h^2$

130. A bullet of mass m is fired into a spherical wooden block of mass M and radius R which is suspended from the ceiling by a cord of length l. If the bullet is completely stopped by the wooden block in a time short compared to the oscillation period of the wooden block, what fraction f of the energy of the bullet is transferred to the kinetic energy of the bullet-block combination?

A) $f = 1$
B) $f = M/m$
C) $f = R/l$
D) $f = mRl/M$
E) $f = m/(m+M)$

131. A string of length l attached to a ball of mass m on a frictionless table passes through a small hole in the table to a mass M. If the ball is put into circular motion with radius r, what is the angular velocity ω with which it needs to move to support the mass M against the acceleration of gravity g?

A) $\omega = \sqrt{Mg/mr}$
B) $\omega = \sqrt{Mg/ml}$
C) $\omega = \sqrt{mg/Mr}$
D) $\omega = \sqrt{mg/Ml}$
E) $\omega = \sqrt{Mg/m(l-r)}$

132. How much rotational energy is contained in the rotating wheels of a bicycle compared to the total energy (rotational plus kinetic) of the entire bicycle? Each wheel has radius r and mass m. The total mass of the bicycle is M and it moves with speed v. You can assume the that entire mass of each wheel is concentrated at the rim.

A) $M/2m$
B) Mr/v
C) Mv/mr
D) $m/(M-2m)$
E) $2m/M$

133. A tethered spacecraft enroute to Mars consists of a nuclear power module of mass m_1 and a living module of mass m_2 connected by a long cable of length d, where d is much larger than the sizes of the two modules. The assembly rotates at an angular rate w to provide artificial gravity g for the astronauts. If the cable is slowly shortened to a length d', what is the new spin rate w'? Assume there are no other forces operating.

A) $w'/w = 1$
B) $w'/w = m_1/(m_1+m_2)$
C) $w'/w = gd/d'$
D) $w'/w = (d/d')^2$
E) $w'/w = d'/d$

134. Let a sphere of radius R and mass M spin with frequency ω. What is the effective gravitational acceleration (i.e., centrifugal plus gravitational) on an object at polar angle θ ($\theta = 0$ at the north pole, $90°$ at the equator, and $180°$ at the south pole)?

A) $a = (GM/R^2)\sin\theta$
B) $a = (GM/R^2) + \omega^2R\cos\theta$

C) $a = (GM/R^2) - \omega^2 R \sin^2\theta$
D) $a = (GM/R^2)\cos^2\theta - \omega^2 R$

135. On a faraway planet, aliens measure the length of a day by the time between two successive high noons, when their sun is at its maximum height in the sky. Assume that their planet's rotational axis is not tilted with respect to the planet's orbital axis, and that the planet orbits in a circle. Let the angular frequency of rotation be $\omega_{\rm rot}$ and the angular frequency of orbital revolution be $\omega_{\rm orb}$, where $\omega < 0$ if the motion is eastward relative to some chosen north pole and $\omega > 0$ if the motion is westward. What is the duration T of the day?

A) $T = 2\pi\,|1/\omega_{\rm rot} - 1/\omega_{\rm orb}|$
B) $T = 2\pi\,|1/\omega_{\rm rot} + 1/\omega_{\rm orb}|$
C) $T = 2\pi\,|1/(\omega_{\rm rot} - \omega_{\rm orb})|$
D) $T = 2\pi\,|1/(\omega_{\rm rot} + \omega_{\rm orb})|$

136. A particle of mass m is attached to a spring and is confined to move on a wire. The fixed end of the spring is a distance l from the nearest part of the wire, and a force F is required to pull the spring to that length l. Assuming that the spring and wire are in a horizontal plane, so that gravity is irrelevant, what is the frequency of small oscillations of the mass along the wire?

A) $\omega = F/m$
B) $\omega = Fml$
C) $\omega = \sqrt{F/(ml)}$
D) $\omega = F/(ml)$

137. Consider a situation similar to the previous problem. Now, however, the particle moves in a circle of radius r. The distance from the fixed end of the spring to the center of the circle is again l, and the force needed to stretch the spring to length l is F. Again ignoring gravity, what is the frequency with which the particle moves around the circle?

A) $\omega = \sqrt{F/(mr)}$
B) $\omega = Fmlr$
C) $\omega = \sqrt{F(r+l)/(mrl)}$
D) $\omega = \sqrt{F(r-l)/(mr)}$

138. Consider a self-gravitating system (e.g., a cluster of stars, or a single star). The virial theorem relates the kinetic energy K, the gravitational potential energy Φ, and the change in the moment of inertia I (units of I are kg m^2). Which of the following could be the virial theorem, assuming a steady background?

A) $\frac{1}{2}d^2I/dt^2 = 2K\Phi$
B) $\frac{1}{2}dI/dt = K + \Phi$
C) $\frac{1}{2}d^2I/dt^2 = 2K + \Phi$
D) $\frac{1}{2}I = K - 2\Phi$

139. Nuclear fusion in stars occurs because nuclei can tunnel through the Coulomb barrier that exists between nuclei. The higher the temperature, or the closer the nuclei, the easier it is to tunnel. Let the matter density be ρ, the hydrogen mass fraction be X, and the temperature be $10^9 T_9$ K. The energy generation rate for the proton-proton reaction can be written

$$\epsilon_{pp} \approx 2.4\rho^\alpha X^\beta T_9^{-2/3}\exp(-3.38T_9^\gamma)\ . \qquad (1)$$

A) Is α positive or negative?
B) Is β positive or negative?
C) Is γ positive or negative?

140. A photon of wavelength λ scatters off of a particle of mass m that is initially at rest. After scattering, the photon travels in a direction that is an angle θ from its initial direction. The final wavelength λ_f of the photon is related to the initial wavelength through the relation

$$\lambda_f - \lambda = \pm\lambda_c(1 - f(\theta))\ , \qquad (2)$$

where λ_c is the Compton wavelength.

A) $\lambda_f - \lambda = [h/(mc)](1 - \cos\theta)$
B) $\lambda_f - \lambda = [h/(mc)](1 - \sin\theta)$
C) $\lambda_f - \lambda = -[h/(mc)](1 - \cos\theta)$
D) $\lambda_f - \lambda = -[h/(mc)](1 - \sin\theta)$

141. Consider the following integral:

$$\int \frac{dx}{x^2\sqrt{x^2 - a^2}}\ . \qquad (3)$$

Let x and a be in units of length. Which of the following could be the value of the integral? Here C is a constant.

A) $\sqrt{x^2 - a^2}/a^2 + C$
B) $\sqrt{x^2 - a^2}/(a^2 x) + C$
C) $\sqrt{x^2 - a^2}/(a^2 x^2) + C$

9. ADVANCED PROBLEMS

Again, a variety of techniques may be needed to rule out the incorrect answers. As always, there may be zero, one, or more correct answers among the selections.

142. Which of the following could be the derivative of the arctangent function?

A) $d\tan^{-1}x/dx = 1/(1 - x^2)$
B) $d\tan^{-1}x/dx = 1/(1 + x^2)$
C) $d\tan^{-1}x/dx = 1/(1 - x^3)$

D) $d\tan^{-1}x/dx = 1/(1+x^3)$

143. The hyperbolic cosecant is defined as sech$y = 2/(e^y + e^{-y})$. Which of the following could be the derivative of its inverse function? Here $0 < x < 1$ and $y \geq 0$.

A) $d\text{sech}^{-1}x/dx = 1/(x\sqrt{1+x^2})$
B) $d\text{sech}^{-1}x/dx = -1/\sqrt{1+x^2}$
C) $d\text{sech}^{-1}x/dx = 1/(x\sqrt{1-x^2})$
D) $d\text{sech}^{-1}x/dx = -1/(x\sqrt{1-x^2})$

144. Two massless springs with spring constants k_1 and k_2 are each attached to a mass m and to a fixed point (springs in parallel). What is the oscillation frequency of the resulting system?

A) $\omega^2 = k_1/m + 2k_2/m$
B) $\omega^2 = (k_1 + k_2)/m$
C) $\omega^2 = 1/(m/k_1 + m/k_2)$
D) $\omega^2 = 2/(m/k_1 + m/k_2)$
E) $\omega^2 = k_1^2/mk_2$

145. One end of a massless spring with spring constant k_1 is attached to a fixed point. The other end is attached to a second massless spring with spring constant k_2 which in turn is connected to a mass m (springs in series). What is the oscillation frequency of the resulting system?

A) $\omega^2 = k_1/m + 2k_2/m$
B) $\omega^2 = (k_1 + k_2)/m$
C) $\omega^2 = 1/(m/k_1 + m/k_2)$
D) $\omega^2 = 2/(m/k_1 + m/k_2)$
E) $\omega^2 = k_1^2/mk_2$

146. Your raging drunk friend sings loudly as he stands barefoot in the thick shag carpet in your living room. As his song comes to an end, he slowly tips forward and starts to fall toward the floor, retaining an amazingly rigid erect posture as he falls. His toes clutch spasmatically as they attempt to maintain their desperate grip on the shag carpet. To minimize the impact his face feels when he hits the floor

A) it is better if his toes grip the carpet.
B) it is better if his toes fail to grip the carpet.
C) it makes no difference if his toes grip or not.

147. In Bohr's semiclassical model of the hydrogen atom, it is assumed that the electron moves in a circle around the nucleus. In the ground state of hydrogen, the angular momentum of the electron equals Planck's constant $\hbar$. The electrostatic potential energy of charges q_1 and q_2 separated by distance r is q_1q_2/r (here we use cgs units; in SI it would be $q_1q_2/(4\pi\epsilon_0 r)$. If an electron of charge e and mass m orbits around a nucleus of charge Ze that is assumed fixed, which of the following could be the binding energy of the atom in its ground state?

A) $E = me^2c/\hbar$
B) $E = Z^2mc^2$
C) $E = Z^2me^4/(2\hbar^2)$
D) $E = me^4/(2Z^2\hbar^2)$

148. A sphere of density ρ and radius R is placed in a fluid of density ρ' with dynamic viscosity η (which has units of kg s^{-1} m^{-1}). If the gravitational acceleration is a constant g, what is the terminal velocity of the sphere if it moves slowly through the fluid? A positive value means it sinks; a negative value means it rises.

A) $v_T = \frac{2}{9}R^2g(\rho - \rho')/\eta$
B) $v_T = \frac{2}{9}\eta/[R(\rho - \rho')]$
C) $v_T = \frac{2}{9}R^2g(\rho' - \rho)$
D) $v_T = \frac{2}{9}\sqrt{Rg}$

149. The equation of hydrostatic equilibrium states that for a fluid in equilibrium in a gravitational field, the gradient in pressure in the fluid opposes the gravitational force on the fluid at all points. If the fluid density is ρ, the gravitational acceleration is a constant g, and the fluid is in a sphere, which of the following could be the equation of hydrostatic equilibrium?

A) $dP/dr = g$
B) $dP/dr = -\rho$
C) $dP/dr = \rho g$
D) $dP/dr = -\rho g$

150. You are designing an amusement park. One of your rides is a drop of a vertical distance h over a horizontal distance l. You design the curve of the track so that the customers fall as quickly as possible from the top to the bottom of the ride. Assuming a uniform gravitational acceleration g, what is the minimum time?

A) $t_{\min} = \left[2(h^2 + l^2)/(gh)\right]^{1/2}$
B) $t_{\min} = \left[2(l^2 + h^2)/(gl)\right]^{1/2}$
C) $t_{\min} = \left[2(h^2 - l^2)/(gh)\right]^{1/2}$
D) $t_{\min} = \left[2(l^2 - h^2)/(gh)\right]^{1/2}$

151. A rod of uniform density and length l is fixed at one end but the other is free to swing. In a gravitational field of constant acceleration g, we want to know the time required for the rod to fall so that the free end is at its lowest point, assuming that the free end starts at an angle $\theta \leq 90^\circ$ from its lowest point. Which of the following expressions have the correct limits for this time?

A) $\sqrt{l/g}\tan\theta$
B) $\sqrt{l/g}\cos\theta$
C) $\sqrt{l/g}(\theta + \theta^3)$
D) $\sqrt{l/g}\sinh\theta$

E) $\sqrt{l/g}\cosh\theta$

152. Someone has discovered a new force that operates between two objects, and depends on some quantity Ψ possessed by each of the objects (for example, Ψ could be the charge or the mass). Establish a spherical coordinate system and define the usual spherical angles θ and ϕ. Assume that the two objects are far from any other objects, and that no other forces are operating. Let the value of Ψ for the first and second objects be Ψ_1 and Ψ_2, respectively. Let the radial and angular locations of the two objects be r_1, θ_1, and ϕ_1, and similarly for the second object. Finally, denote the distance between the two objects by d. What are possible expressions for the magnitude of the new force? Here a positive force means an attraction, a negative force means a repulsion.

A) $F = (Gc/\hbar)(\cos\Psi_1 + \sin\Psi_2)$
B) $F = \Psi_1\Psi_2(\sin\theta_1 + \sin\theta_2)$
C) $F = \Psi_1^2\Psi_2$
D) $F = \sin^2[(\Psi_1+\Psi_2)/\sqrt{\Psi_1^2+\Psi_2^2}]/d^2$
E) $F = \Psi_1\Psi_2\sin(\phi_1-\phi_2)$

153. A triangle has sides of length s_1, s_2, and s_3. Define $p \equiv s_1s_2s_3$ and $d \equiv \frac{1}{2}(s_1+s_2+s_3)$. What is the area A of the triangle?

A) $A = \sqrt{ps_1}$
B) $A = \sqrt{pd}$
C) $A = \sqrt{d(d-s_1)(d-s_2)(d-s_3)}$
D) $A = (d-s_1)(d+s_2)$

154. Consider a spherical triangle, that is, a triangle drawn on a sphere whose sides are segments of great circles. Let the interior angles of the triangle be A, B, and C, and let the angles on the sphere subtended by the arcs opposite the angles be a, b, and c. Which of the following could be valid relations between the angles?

A) $\cos A/\cos a = \cos B/\cos b = \cos C/\cos c$
B) $\sin A/\sin a = \sin B/\sin b = \sin C/\sin c$
C) $\sin A\sin a = \sin B\cos b = \cos C\cos c$
D) $\sin a = \cos b\cos c + \sin b\sin c\sin A$
E) $\cos a = \cos b\cos c + \sin b\sin c\cos A$
F) $\cos a = \cos b\cos c + \sin b\sin c\sin A$

155. This number of faces f, edges e, and vertices v of a solid with planar faces satisfies which of the following equations?

A) $v+e+f = 26$
B) $fv = e+2$
C) $fv = 4e$
D) $v+f = e+2$
E) $v+e-f = 0$
F) $fv = 2(e+2)$

156. A uniform stick of length L and mass M lying at rest on a frictionless surface is hit by a bullet of mass m moving at velocity v. If the projectile strikes the stick a distance $h > 0$ from the center and is embedded in it, what is the final velocity v_{cm} of the stick and bullet combination?

A) $v_{cm} = mv/(m+M)$
B) $v_{cm} = mv(h/L)/(m+M)$
C) $v_{cm} = mvh/L$
D) $v_{cm} = mvh/ML$
E) $v_{cm} = mv/M$

157. In the above problem, what is the distance x between the center of the stick and the new center of mass?

A) $x = 0$
B) $x = h$
C) $x = hm/M$
D) $x = mL/(M+m)$
E) $x = mh/(2M+m)$

158. In the above problem, what is the moment of inertia of the stick bullet combination about its center of mass? Use the fact that the moment of inertia of a point mass m a distance r from the rotation axis is $I = mr^2$ and that a stick of length L about its center of mass has $I = ML^2/12$.

A) $I = (x-h)^2m + M[(L/2+x)^3 + (L/2-x)^3]/(3L)$
B) $I = (x-h)^2m + M[(L/2+x)^2 + (L/2-x)^2]/(3L)$
C) $I = (x+h)^2m + M[(L/2+x)^3 + (L/2-x)^3]/(2L)$
D) $I = (x+h)^2m + M[(L/2+x)^3 - (L/2-x)^3]/(3L)$

159. In the above problem, what is the final angular speed of the stick bullet combination about its new center of mass in terms of the moment of inertia I about the center of mass?

A) $w = mvh/I$
B) $w = mvL/I$
C) $w = m^2vh/[I(m+M)]$
D) $w = mv/I$
E) $w = MvL/I$

160. A projectile is launched from the surface of a sphere of mass M and radius R, which is rotating with angular velocity ω. The projectile is initially at a polar angle θ and an azimuthal angle ϕ, and is launched horizontally, but at an angle α relative to the local rotational direction (e.g., $\alpha = 0$ is along the velocity of rotation). If air resistance is neglected, the initial velocity v_{esc} required to just escape from the sphere may be written as

A) $v_{\text{esc}} = \sqrt{2GM/R - \omega^2R^2\sin^2\theta\sin^2\alpha} + \omega R\sin\theta\cos\alpha$
B) $v_{\text{esc}} = \sqrt{2GM/R - \omega^2R^2\sin^2\theta\sin^2\alpha} - \omega R\sin\theta\cos\alpha$

C) $v_{\rm esc} = \sqrt{2GM/R - \omega^2 R^2 \cos^2\theta \sin^2\alpha} - \omega R \cos\theta \cos\alpha$
D) $v_{\rm esc} = \sqrt{2GM/R + \omega^2 R^2 \sin^2\theta \sin^2\alpha} - \omega R \sin\theta \cos\alpha$

161. A block of mass m is tied to a post by means of a spring of spring constant k. It is made to rotate around the post in a horizontal plane at an angular frequency ω. We neglect friction with the table and air resistance. If the spring at rest has length l_0, what is its length when the block is rotating?

A) $l = l_0/(1 + m\omega^2/k)$
B) $l = l_0/(2 - m\omega^2/k)$
C) $l = l_0 \left[1 + \omega/\left(Gm/l_0^3\right)^{1/2}\right]$
D) $l = l_0/(1 - m\omega^2/k)$

162. Consider two diatomic molecules. The molecules are identical with each other (i.e., their atoms are the same) except that the isotopes of the atoms in the first molecule differ from those in the second. If the masses of the atoms in the first molecule are m_1 and m_2, and in the second are m_1' and m_2', what is the ratio of the oscillation frequencies ω and ω' of the molecules? Note that the "spring constants" of the two molecules are equal to each other.

A) $\omega'/\omega = \sqrt{m_1 m_1'/(m_2 m_2')}$
B) $\omega'/\omega = \sqrt{m_1 m_2}$
C) $\omega'/\omega = \sqrt{m_1 m_2 (m_1' - m_2')/[m_1' m_2' (m_1 + m_2)]}$
D) $\omega'/\omega = \sqrt{m_1 m_2 (m_1' + m_2')/[m_1' m_2' (m_1 + m_2)]}$

163. A solid object on the surface of a fluid of kinematic viscosity ν (dimensions m^2 s^{-1}) oscillates with frequency ω. The fluid oscillates with the object, but the amplitude of the oscillation becomes exponentially small far from the object. What is the characteristic depth δ to which the oscillation penetrates?

A) $\delta = \nu\omega$
B) $\delta = \sqrt{2\nu\omega}$
C) $\delta = \sqrt{\omega/\nu}$
D) $\delta = \sqrt{2\nu/\omega}$

In fluid mechanics, there are several dimensionless numbers that are often used to characterize the flow or physics of the fluid. The next few problems deal with the definition of these numbers. All of the following are pure numbers.

164. The Reynolds number $\mathcal{R}$ depends on the kinematic viscosity ν of the fluid (units of m^2 s^{-1}), the size l of the object, and the velocity v of the object through the fluid. A low Reynolds number means the flow is slow and smooth; a high Reynolds number means the flow is fast and turbulent. Which of the following could be the definition of $\mathcal{R}$?

A) $\mathcal{R} = \nu l v$
B) $\mathcal{R} = \nu/lv$
C) $\mathcal{R} = 1/(\nu l v)$
D) $\mathcal{R} = vl/\nu$

165. The Strouhal number $\mathcal{S}$ is used when flows are not steady. It therefore depends on the size l of the object and its velocity v through the fluid, and also on the time scale τ over which the flow changes. It is smaller for rapid change. Which of the following could be the definition of $\mathcal{S}$?

A) $\mathcal{S} = v\tau/l$
B) $\mathcal{S} = vl/\tau$
C) $\mathcal{S} = l/(v\tau)$
D) $\mathcal{S} = 1/(vl\tau)$

166. The Prandtl number $\mathcal{P}$ depends on the kinematic viscosity ν (as above) and the "thermometric conductivity" χ (units of m^2 s^{-1}). The Prandtl number is small when heat conduction dominates over viscous effects. Which of the following could be the definition of $\mathcal{P}$?

A) $\mathcal{P} = \nu\chi$
B) $\mathcal{P} = \nu/\chi$
C) $\mathcal{P} = \chi/\nu$
D) $\mathcal{P} = 1/(\nu\chi)$

167. A sound wave travels from one medium to another. In the first medium the sound speed is c_1 and the angle of propagation of the wave relative to the interface between the media is θ_1. In the second medium the sound speed is c_2 and the angle of propagation is θ_2. Which of the following could be a correct relation?

A) $\sin\theta_1/\sin\theta_2 = c_1 c_2$
B) $\cos\theta_1/\cos\theta_2 = c_1/c_2$
C) $\sin\theta_1 \cos\theta_2 = c_2/c_1$
D) $\sin\theta_1/\sin\theta_2 = c_1/c_2$

168. A turbulent fluid emits sound waves. If the turbulent velocity is v, the sound speed in the fluid is c_s, and the length over which turbulent fluctuation velocities is correlated is l, what is the energy per time ϵ_s emitted as sound by unit mass of the turbulent fluid?

A) $\epsilon_s \propto c_s^6/(v^3 l)$
B) $\epsilon_s \propto c_s^2 v l$
C) $\epsilon_s \propto v^8/(c_s^5 l)$
D) $\epsilon_s \propto v^3/l$

169. When matter burns slowly (e.g., a lit piece of paper), there is a thin "combustion zone" over which the flame is active. Let τ be the time for which the reaction lasts at a given point, and let χ (m^2 s^{-1}) be the thermometric conductivity. Which of the following could be true of the thickness δ of the flame front?

A) $\delta \propto \chi/\tau$
B) $\delta \propto \tau/\chi$
C) $\delta \propto \sqrt{\chi\tau}$
D) $\delta \propto 1/\sqrt{\chi\tau}$

170. A molecule with moment of inertia I can only undergo a rotational transition if the temperature is high enough. What is that critical temperature? Here k is the Stefan-Boltzmann constant, so that thermal energy is kT.

A) $T_{\rm crit} = \hbar^2/(kI)$
B) $T_{\rm crit} = \hbar kI$
C) $T_{\rm crit} = I/k$
D) $T_{\rm crit} = kI/\hbar$

171. You launch a rocket straight up from the Earth's North pole and it rises up then falls back to Earth. The maximum height above Earth's surface H is given by one of the expressions below. Here R_E is the Earth's radius, $X = v^2 R_E/GM_E$, G is the gravitational constant, M_E is the Earth's mass and v is the launch velocity. Rule out as many of the following expressions as you can.

A) $H = R_E X/(1+\sqrt{X})$
B) $H = R_E X/(1-X)$
C) $H = R_E X/(2-X)$
D) $H = R_E(1-X)/(2-X)$
E) $H = vX^2/(2-X)$
F) $H = R_E X/2$
G) $H = R_E X^2/(2-X)$
H) $H = R_E X|1-X|/(2-X)$

172. A mass m is free to slide along a slot in a frictionless table. The mass is attached to a fixed point a distance l directly under the slot by a spring of spring constant k, and equilibrium length l_0. If the system is allowed to oscillate, what will be the frequency of small oscillations?

A) $\omega^2 = k/m$
B) $\omega^2 = (l/l_0)k/m$
C) $\omega^2 = (1 + l/l_0)k/m$
D) $\omega^2 = (1 + l_0/l)k/m$
E) $\omega^2 = (l_0/l)k/m$

173. Suppose we have a system with a star of mass M_1 and a planet of mass $M_2 \ll M_1$ orbiting the star at a separation R. Consider a test mass $m \ll M_2$ in the system, and assume that it and both larger masses are orbiting in circles at an angular frequency $\Omega = \sqrt{G(M_1+M_2)/R^3}$. The first Lagrange point L_1, where the total radial force (gravitational plus centrifugal) on the test mass vanishes, is between M_1 and M_2. To lowest order, what is the distance of L_1 from the center of mass of the system?

A) $d = R[1 - (M_2/3M_1)^{1/3}]$
B) $d = R(M_1 + M_2)/M_1$
C) $d = R[1 - (M_1/5M_2)]$
D) $d = RM_2/(M_1 + M_2)$

174. Two equal-mass particles have a perfectly elastic head-on collision. The initial speeds of particle 1 and particle 2 were v_1 and v_2, respectively. After the collision, they travel at new speeds v_1' and v_2', at angles θ_1 and θ_2 relative to their original directions. Which of the following relations must hold?

A) $v_1' \sin\theta_1 = v_2' \sin\theta_2$
B) $v_1' \sin\theta_1 = v_2' \cos\theta_2$
C) $v_1'^2 \cos^2\theta_1 + v_2'^2 \cos^2\theta_2 = v_1^2 \sin^2\theta_1 + v_2^2 \sin^2\theta_2$
D) $v_1'^2 + v_2'^2 = v_1^2 + v_2^2$

175. The Heisenberg uncertainty principle states that the location and momentum of a particle cannot be known simultaneously with perfect precision. In a very dense environment the particles are well-localized, hence their momenta are correspondingly large. Associated with this momentum is an energy, the Fermi energy E_F. In the nonrelativistic limit, what is the Fermi energy per particle if they have mass m and number density n?

A) $E_F \propto \hbar^2 n^{1/3}/m$
B) $E_F \propto \hbar^2 n^{2/3}/m$
C) $E_F \propto \hbar^{-2} n/m$
D) $E_F \propto nm$

176. In the previous problem, what is the Fermi energy in the relativistic limit? here c is the speed of light.

A) $E_F \propto \hbar n^{1/3} c$
B) $E_F \propto \hbar^2 n^{2/3}/c$
C) $E_F \propto \hbar^{-2} nc$
D) $E_F \propto nmc$

10. MASTER'S CHALLENGE

These problems also involve a combination of units, limits, symmetry, and so on. Here, however, we've broadened the topics to include general relativity, fluid mechanics, quantum mechanics, and other areas of physics. If you can rule out the incorrect answers in these problems, you have mastered the fundamentals and have taken a big step towards intuition in physics!

177. An oblate spheroid is formed by rotating an ellipse around its minor axis (i.e., a squashed sphere). Suppose a is the major semiaxis, b is the minor semiaxis, and e is the eccentricity of the ellipse. What is the surface area of the oblate spheroid?

A) $S = \pi ab$
B) $S = 2\pi a^2 + \pi(b^2/e)\ln[(1+e)/(1-e)]$
C) $S = 2\pi a^2 + \pi(b^2/e)\ln[(1-e)/(1+e)]$
D) $S = 2\pi b^2 + 2\pi(ab/e)\sin^{-1} e$
E) $S = \pi b^2 + 2\pi(ab/e)\sin^{-1} e$

178. A prolate spheroid is formed by rotating an ellipse around its major axis (i.e., a stretched sphere). Suppose a is the semimajor axis, b is the semiminor axis, and e is the eccentricity of the ellipse. What is the surface area of the prolate spheroid?

A) $S = \pi ab$
B) $S = 2\pi a^2 + \pi(b^2/e)\ln[(1+e)/(1-e)]$
C) $S = 2\pi a^2 + \pi(b^2/e)\ln[(1-e)/(1+e)]$
D) $S = 2\pi b^2 + 2\pi(ab/e)\sin^{-1} e$
E) $S = \pi b^2 + 2\pi(ab/e)\sin^{-1} e$

179. The relativistically-correct expression for adding two velocities is:

A) $v_{total} = (v_1 + v_2)/(1 + v_1 v_2/c^2)$
B) $v_{total} = (v_1 + v_2)/(1 - v_1 v_2/c^2)$
C) $v_{total} = (v_1 + v_2)/(1 + v_1^2/c^2)$
D) $v_{total} = (v_1^2 + v_2^2)/c(1 + v_1 v_2/c^2)$
E) $v_{total} = (v_1 - v_2)/(1 + v_1 v_2/c^2)$

180. Two point masses, of mass m_1 and m_2, are orbiting each other in circular orbits. The separation between the two is a and we define the total mass $M = m_1 + m_2$ and the reduced mass $\mu = m_1 m_2/M$. To lowest order, what is the time scale on which they will spiral in to each other due to gravitational radiation?

A) $T = (256/5)[G^3\mu M^2/(c^7 a^2)]$
B) $T = (5/256)[c^5 a^4/(G^3\mu M^2)]$
C) $T = (256/5)(GM/c^3)\exp[-GM/ac^2]$
D) $T = (5/256)[c^5 a^4/(G^3 m_1 m_2^2)]$

181. A planet of mass m orbits a star of mass $M \gg m$ in an ellipse with semimajor axis a and eccentricity e. In radians, what is the magnitude $\delta\phi$ of the angular perihelion shift per orbit due to general relativity, assuming $\delta\phi \ll 1$?

A) $\delta\phi = 6\pi Gm/[ac^2(1-e^2)]$
B) $\delta\phi = 6\pi GM/[ac^2(1-2e^2)]$
C) $\delta\phi = 6\pi GM/[ac^2(1-e^2)]$
D) $\delta\phi = 6\pi M/[m(1-e^2)]$

182. A wave of wavelength λ (and hence wavenumber $k = 2\pi/\lambda$) and small amplitude travels over the surface of a lake of depth h. The gravitational acceleration is uniform, with value g. What is the speed v of the wave? Ignore surface tension and capillary effects. Recall that $\tanh x = (e^x - e^{-x})/(e^x + e^{-x})$ and $\cosh x = (e^x + e^{-x})/2$.

A) $v = \frac{1}{2}\sqrt{k\left[\tanh kh + kh/\cosh^2 kh\right]/(g\tanh kh)}$
B) $v = \frac{1}{2}\sqrt{g\left[\tanh kh + kh/\cosh^2 kh\right]/(k\tanh kh)}$
C) $v = \frac{1}{2}\sqrt{g\left[\cosh kh + kh/\tanh^2 kh\right]/(k\tanh kh)}$
D) $v = \frac{1}{2}\sqrt{g\left[\tanh kh + (\cosh^2 kh)/kh\right]/(k\tanh kh)}$

183. A thin disk is placed in a fluid that has density ρ kg m^{-3} and kinematic viscosity ν m^2 s^{-1}. The disk has radius R and angular frequency Ω. What is the torque exerted on the disk by the fluid?

A) $N \approx 2R^5\rho\Omega^2$
B) $N \approx 2R\Omega\nu$
C) $N \approx 2\nu^3/(R\Omega)$
D) $N \approx 2R^4\rho\sqrt{\nu\Omega^3}$

184. An object of size l moves through a fluid. The Reynolds number of the motion is $\mathcal{R}$. Fluid close to the object moves with the object; fluid far away is undisturbed. The effect of the object dies away on a characteristic distance δ, which is the thickness of the boundary layer. What could δ be?

A) $\delta \propto l/\sqrt{\mathcal{R}}$
B) $\delta \propto l$
C) $\delta \propto l^2$
D) $\delta \propto l\sqrt{\mathcal{R}}$

185. A thermometer placed in a moving fluid measures a different temperature than measured by a thermometer moving with the fluid. The temperature difference ΔT depends on the speed v of the fluid, the specific heat per unit mass c_P (which has units of J kg^{-1} K^{-1}), the viscosity, and the thermometric conductivity. Which of the following could be a correct expression for the temperature difference?

A) $\Delta T \propto \mathcal{R}v^2/c_P$
B) $\Delta T \propto \mathcal{P}c_P/v^2$
C) $\Delta T \propto \mathcal{P}v^2/c_P$
D) $\Delta T \propto \mathcal{S}c_P v^2$

186. A sphere oscillates radially (i.e., it expands and contracts) in a fluid of density ρ and sound speed c_s. Which of the following could be true about the intensity (energy per time) of the sound waves produced by the oscillation? Here V is the instantaneous volume of the sphere.

A) $I = \rho(dV/dt)c_s^2/4\pi$
B) $I = \rho(d^2V/dt^2)^2/(4\pi c_s)$
C) $I = \rho c_s^8/(4\pi d^5V/dt^5)$
D) $I = \rho(d^2V/dt^2)c_s/4\pi$

187. A block of mass m is tied to a post by means of a spring of spring constant k. It is made to rotate in a *vertical* plane at an angular frequency ω. Air resistance is neglected, and we assume constant gravitational acceleration g. Let the angular location of the block be ϕ, where $\phi = 0$ at the top of the cycle and ϕ increases counterclockwise. We assume that the spring is always at its equilibrium length at a given angle, and does not oscillate around that length. If the spring at rest has length l_0, what is its length when the block is rotating?

A) $l = [l_0 - (mg/k)\cos\phi]/(1 - m\omega^2/k)$
B) $l = [l_0 + (mg/k)\cos\phi]/(1 - m\omega^2/k)$
C) $l = [l_0 - (mg/k)\sin\phi]/(1 - m\omega^2/k)$
D) $l = [l_0 - (mg/k)\cos\phi]/(1 - gm/kl_0)$

188. What is the frequency of in-phase oscillations of a double pendulum where a bob of mass m_1 is attached to the ceiling by a string of length l_1, and a bob of mass m_2 is attached to m_1 by a string of length l_2? There is a constant gravitational acceleration g.

A) $\omega^2 = gm_1/(m_1l_1 + m_2l_2)$
B) $\omega^2 = g/(l_1 + l_2)$
C) $\omega^2 = g/(l_1 + l_2)(1 + l_1m_1/l_2m_2)$
D) $\omega^2 = g/(l_1 + l_2)(1 + l_1l_2m_1/(l_1^2m_1 + l_2^2m_2))$
E) $\omega^2 = g(m_1 + m_2)/(m_1l_1 + m_2(l_1 + l_2))$

189. Two pendulums of mass m are attached to the ceiling by strings of length l. The attachment points are a distance l apart, and the pendulum bobs are attached by a spring of equilibrium length l and spring constant k. One normal mode of oscillation has the pendulums oscillating in phase. There is a constant gravitational acceleration g. This mode's frequency is:

A) $\omega^2 = g/l$
B) $\omega^2 = \sqrt{gk/lm}$
C) $\omega^2 = g/l + k/(2m)$
D) $\omega^2 = g/l + k/m$
E) $\omega^2 = k/m$

190. The other normal mode of oscillation for the above problem has the pendulums oscillating 180 degrees out of phase. This mode's frequency is:

A) $\omega^2 = g/l$
B) $\omega^2 = \sqrt{gk/lm}$
C) $\omega^2 = g/l + k/(2m)$
D) $\omega^2 = \sqrt{g^2/l^2 + k^2/m^2}$
E) $\omega^2 = k/m$
F) $\omega^2 = g/l - k/(2m)$

191. A pendulum consisting of a massless string of length l and a pointlike plumb bob of mass m is set in small-amplitude motion, such that its angular deviation from the local vertical is always small. It is in a planetarium which is at a polar angle θ and an azimuthal angle ϕ on the Earth (where the azimuthal angle is measured eastward from Greenwich). It is carefully set swinging in a plane at a time t. As Foucault showed, an observer fixed to the Earth will see the plane of oscillation precess, with a period P. Assuming that the Earth is a sphere of radius R_E rotating with a period P_0, what is the Foucault precession period P?

A) $P = P_0/|\cos\theta\sin\phi|$
B) $P = (P_0^2/t)/\cos^2\theta$
C) $P = P_0/\sin\theta$
D) $P = P_0/|\cos\theta|$
E) $P = (l/R_E)P_0/|\cos\theta|$
F) $P = P_0\sin\theta$

192. Define a coordinate system K in which the z-axis points upwards. Consider a reference frame K' which is moving along the z axis with a velocity $v = \beta c$ (positive v means movement along the positive z axis). In the K' frame, there is an object moving with velocity u' in a direction that is an angle θ' from the z=z' axis, as measured in the K' frame. Define $\gamma \equiv 1/\sqrt{1-\beta^2}$. In the K frame, could the angle θ to the z axis made by the velocity of the object be given by:

A) $\cos\theta = u'\cos\theta'/[\gamma(u'\cos\theta' + v)]$
B) $\tan\theta = u'\sin\theta'/[\gamma(u'\cos\theta' + v)]$
C) $\cot\theta = u'\cos\theta'/[(\gamma - 1)(u'\sin\theta' + v)]$
D) $\tan\theta = u'\sin\theta'/[\gamma(u'\cos\theta' + \beta v)]$

193. A person stands at polar angle θ on the Earth, where $\theta = 0$ at the north pole, $\theta = 90^\circ$ at the equator, and $\theta = 180^\circ$ at the south pole. The Earth is assumed to be a sphere of radius R_E rotating uniformly at an angular velocity Ω from west to east. The person throws a ball, horizontally, to a friend. The throw is an angle ψ from due south, measured counterclockwise (i.e., an eastward throw has $\psi = 90^\circ$, a northward throw has $\psi = 180^\circ$, and so on). The ball has speed v and is in the air for a time T, at which point it reaches the friend's polar angle. What is the distance by which the Coriolis force causes the ball to drift eastward relative to the friend after time T (a negative distance indicates a westward drift)? Assume that the ball does not cross the equator during the throw. The distance by which the Coriolis force causes the ball to drift eastward relative to the friend after time T may be represented as d, where a negative distance indicates a westward drift.

A) $d = \frac{1}{2}\Omega vT^2\cos\theta\cos\psi$
B) $d = \frac{1}{2}\Omega vT^2\sin\theta\cos\psi$
C) $d = \frac{1}{2}\Omega vT^2\cos\theta\sin\psi$
D) $d = \frac{1}{2}\Omega vT^2\sin\theta\sin\psi$
E) $d = -\frac{1}{2}\Omega vT^2\cos\theta\cos\psi$
F) $d = -\frac{1}{2}\Omega vT^2\sin\theta\cos\psi$

G) $d = -\frac{1}{2}\Omega v T^2 \cos\theta \sin\psi$
H) $d = -\frac{1}{2}\Omega v T^2 \sin\theta \sin\psi$

194. Consider a black hole of angular momentum J and mass M. Its rotation means that spacetime is dragged in the direction of the hole's rotation; for example, a particle dropped from rest at infinity radially towards the black hole will, at a radius r, appear to have an angular velocity ω relative to a distant observer. To first order in J, what is that angular velocity?

A) $\omega = c^3 r^3/2GJ$
B) $\omega = (e^2/\hbar c)c^3/GM$
C) $\omega = 2GJ/c^2 r^3$
D) $\omega = J/Mr^2 + (c^3/GM)(GM/rc^2)^2$

195. Let a black hole have mass M and angular momentum J, and define the dimensionless angular momentum $j \equiv cJ/GM^2$. Black holes can have $|j|$ between 0 and 1, where $j > 0$ means clockwise as seen from above the black hole and $j < 0$ means counterclockwise. A test particle (i.e., mass $m \ll M$) is orbiting the hole in a clockwise circular orbit of radius r, in the rotational plane. An observer at a great distance notes the orbital frequency ω. Which of the following expressions could be correct? Note that a rotating black hole drags spacetime in the direction of its rotation, but that at large distances this has diminishing effect.

A) $\omega = \sqrt{GM/r^3(1-j)}$
B) $\omega = \sqrt{GM/r^3} + 2j^2G^2M^2/r^3c^3$
C) $\omega = \sqrt{GM/r^3}(1+j)$
D) $\omega = \sqrt{GM/r^3}/\left[1 + j(GM/rc^2)^{3/2}\right]$

196. A particle of mass m is placed in an infinitely deep square well with width a (units of meters). Quantum mechanically, the only allowed wavefunctions are sine waves, $\psi(x) \propto \sin(n\pi x/a)$, where n is a positive integer. The smaller the wavelength of this wave function is, the larger is its energy. Which of the following could be the energy of the nth wave function?

A) $E_n = [\pi^2\hbar^2/(2ma^2)]n^2$
B) $E_n = [\pi^2 m^2 c^3 a/\hbar]n^2$
C) $E_n = [\pi^2\hbar/(3ma)]n^{-2}$
D) $E_n = [\pi^2\hbar^2/(2ma^2)]n^{-2}$

197. A particle of mass m is placed in a rectangular box of dimension a along the x-axis, b along the y-axis, and c along the z-axis. If the x-component of the wavefunction is $\psi(x) \propto \sin(n\pi x/a)$, what is the pressure exerted by the particle on the sides of the box perpendicular to the x-axis?

A) $P = \pi^2\hbar^2 n^{-2}/(ma^3bc)$
B) $P = \pi^2\hbar^2 n^2/(mabc)$
C) $P = \pi^2\hbar^2 n^{-2}/(mabc^3)$
D) $P = \pi^2\hbar^2 n^2/(ma^3bc)$
E) $P = \pi^2\hbar^2 n^2/(mb^2c^3)$

198. Many derivations in quantum mechanics can be checked by letting $\hbar \longrightarrow 0$ and making sure that the answer corresponds to the classical limit. Consider a simple harmonic oscillator, which has a frequency $\omega = 2\pi\nu$ and mass m. Which of the following could be the possible energy states for the oscillator? Here n is a nonzero integer, $0, 1, 2, \ldots$.

A) $E_n = \frac{1}{2}\hbar\omega$
B) $E_n = (n+1/2)\hbar\omega$
C) $E_n = (n+1/2)m^2c^4/(\hbar\omega)$
D) $E_n = n\omega$

199. de Broglie found that, quantum mechanically, a particle with momentum p acts like a wave with a certain wavelength. What is this wavelength?

A) $\lambda = 2\pi\hbar p$
B) $\lambda = 2\pi/(\hbar p)$
C) $\lambda = 2\pi p/\hbar$
D) $\lambda = 2\pi\hbar/p$

200. The Heisenberg uncertainty principle can be phrased in terms of an uncertainty in energy ΔE measured in a time Δt. Which of the following could be the statement of this principle?

A) $\Delta E \Delta t \geq \hbar/2$
B) $\Delta E \Delta t \geq 2/\hbar$
C) $\Delta E/\Delta t \geq 2/\hbar$
D) $\Delta E \Delta t \leq \hbar/2$
E) $\Delta t/\Delta E \leq \hbar/2$

201. A compact object moving through space can accrete matter from the surrounding medium. If the medium is cold and an object of mass M moves with velocity v through a medium of density ρ, gas that moves close enough that the escape velocity from the object exceeds v will be captured. Which of the following could be the accretion rate $\dot{M}$?

A) $\dot{M} \propto (GM)^2\rho v^{-3}$
B) $\dot{M} \propto (GM)^2\rho v^3$
C) $\dot{M} \propto (GM)^{-2}\rho v^{-3}$
D) $\dot{M} \propto (GM)^{-2}v^3$